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# Derivatives, forms and vector fields on the $\kappa$-deformed Euclidean space

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**Abstract**

The model of $\kappa$-deformed space is an interesting example of a noncommutative space, since it allows a deformed symmetry. In this paper, we present new results concerning different sets of derivatives on the coordinate algebra of $\kappa$-deformed Euclidean space. We introduce a differential calculus with two interesting sets of one-forms and higher-order forms. The transformation law of vector fields is constructed in accordance with the transformation behaviour of derivatives. The crucial property of the different derivatives, forms and vector fields is that in an $n$-dimensional spacetime there are always $n$ of them. This is the key difference with respect to conventional approaches, in which the differential calculus is $(n + 1)$-dimensional. This work shows that derivative-valued quantities such as derivative-valued vector fields appear in a generic way on noncommutative spaces.

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## 1. Introduction

The model of $\kappa$-deformed spacetime was originally introduced in [1, 2] (for a more comprehensive list of references, see [3]) and has afterwards been discussed by several groups, both from a mathematical and a physical perspective.

The main interest in the $\kappa$-deformed spacetime comes from the fact that this model is a mild deformation of spacetime. There is only one coordinate that does not commute with all others. Therefore this is a sufficiently simple model that may serve as a playground to develop generic concepts for noncommutative spaces. The reason is that it is a mathematically consistent deformation of spacetime, compatible with a simultaneous deformation of the symmetry structure, the $\kappa$-Poincaré algebra. Recently, a new motivation for studying the $\kappa$-deformed spacetime has appeared. Namely, it is a well-founded framework for so-called doubly special relativity, i.e. special relativity with a second invariant scale (cf [4, 5]).

In its Minkowski version the $\kappa$-deformed spacetime has first been treated as the translational sector of the $\kappa$-Poincaré group. It contains one distinguished coordinate which does not commute with all other coordinates. The $\kappa$-Poincaré group as a Hopf algebra is dual to the $\kappa$-Poincaré algebra [6], which has first been derived contracting the Hopf algebra $SO_q(3, 2)$. It has been found that in the so-called bicrossproduct basis [7] the generators of the $\kappa$-Lorentz algebra fulfil the commutation relations of the undeformed Lorentz Lie algebra. Nevertheless, the symmetry generators of the $\kappa$-Poincaré Hopf algebra act in a deformed way on products of functions.

The bicrossproduct basis of the $\kappa$-Poincaré algebra can be obtained by a constructive procedure requiring consistency with the algebra of coordinates. The transformation behaviour of the coordinates under rotations leads to an undeformed algebra of rotations. Consistency with this undeformed algebra of rotations and consistency with the algebra of coordinates are then the two touchstones to construct geometrical concepts such as derivatives, forms and vector fields. This is the content of this paper. It complements several analyses concerning the $\kappa$-deformed spacetime which we recently presented in [3, 8]. Our construction of a field theory based on representing quantities of $\kappa$-deformed spacetime by means of $\star$-products is similar to other recent approaches [9, 10].

This work is organized as follows: in the subsequent section we fix our notation concerning the abstract algebra, recapitulating results of [3]. We present the Hopf algebra properties of our model in an introductory way. Several sets of derivatives are discussed; they either have simple commutation relations with the coordinates or simple transformation behaviour under rotations.

In section 3 we discuss three $\star$-products which can be defined in a generic way. A closed formula for the symmetric $\star$-product is derived. The symmetry generators are represented in terms of derivative operators both for the symmetric $\star$-product and for normal ordered $\star$-products.

In section 4 we present an analysis of the differential calculus on the $\kappa$-deformed space. In contrast to results in the literature we argue that an $n$-dimensional $\kappa$-deformed space can be equipped with an $n$-dimensional differential calculus of one-forms. To be able to do so, we have to accept that the commutation relations of one-forms with coordinates become derivative valued. We calculate frame one-forms, which commute with the coordinates, and construct representations of the forms, regarding them as derivative-valued maps in the algebra of functions of commuting variables.

In section 5 we introduce vector fields by generalizing the transformation behaviour of derivatives under the $\kappa$-deformed rotations. We construct maps between different vector fields and find that generically also vector fields are derivative-valued quantities.

## 2. Derivatives on the $\kappa$-deformed space

### *2.1. The $\kappa$-deformed space*

The $n$-dimensional $\kappa$-deformed space is a noncommutative space of the Lie algebra type, i.e. it is the associative factor space algebra $\mathcal{A}_{\hat{x}}$ freely generated by $n$ abstract coordinates $\hat{x}^\mu$, divided by the ideal freely generated by the commutation relations [11]

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_{\lambda}\hat{x}^\lambda. \tag{1}$$

The structure constants for the $\kappa$-deformed space are $C_\lambda^{\mu\nu} = a^\mu \delta_\lambda^\nu - a^\nu \delta_\lambda^\mu$. The characteristic deformation vector $a^\mu$ can be rotated without loss of generality into one of the $n$ directions $a^\mu = a\delta_n^\mu$. In this case there is one special coordinate $\hat{x}^n$, which does not commute with all other coordinates

$$[\hat{x}^n, \hat{x}^j] = \mathrm{i}a\hat{x}^j, \qquad [\hat{x}^i, \hat{x}^j] = 0, \qquad i, j = 1, 2, \ldots, n-1. \tag{2}$$

Coordinates without hat such as $x^\mu$ denote ordinary variables. In most discussions of $\kappa$-deformed spacetime, $\hat{x}^n$ is taken to be the time coordinate of a four-dimensional Minkowski spacetime. The restriction to four dimensions had already been lifted in [12]. Identifying $\hat{x}^n$ with the direction of time is an additional choice. In our approach $\hat{x}^n$ is an arbitrary direction of an $n$-dimensional Euclidean space. This setting is chosen for transparency of the calculus. Indices can arbitrarily be lowered or raised with a formal metric $g^{\mu\nu} = \delta^{\mu\nu}$. Greek indices take values $1, \ldots, n$, while Roman indices apart from $n$ run from $1, \ldots, n-1$. We use the Einstein summation conventions.

Our formulae generalize in a straightforward way to a Minkowski setting [8]. Also for non-diagonal metrics there are generalizations [13]. We use the notation (2) instead of $\kappa = \frac{1}{a}$, since this is more convenient for working in configuration space.

Derivatives are regarded as maps in the coordinate algebra, $\widehat{\partial}_\mu : \mathcal{A}_{\hat{x}} \to \mathcal{A}_{\hat{x}}$. In order to define derivatives $\widehat{\partial}_\mu$, we demand that they

- are consistent with (2);
- are a deformation of ordinary derivatives, i.e. $[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu + \mathcal{O}(a)$;
- commute among themselves.

These restrictions on derivatives $\widehat{\partial}_\mu$ are weak, there exists a wide range of possible solutions

$$[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu + \sum_j a^j (\widehat{\partial}_{(\mu,\nu)})^j. \tag{3}$$

The symbolic notation denotes all terms of a power series expansion in the derivatives $\widehat{\partial}_\mu$, which are consistent with the index structure.

With the additional condition that the commutator $[\widehat{\partial}_\mu, \hat{x}^\nu]$ is linear in the derivatives, there are three one-parameter families of solutions:

$$\begin{aligned}
&\left[\hat{\partial}_n^{c_1}, \hat{x}^n\right] = 1 + \mathrm{i}ac_1\hat{\partial}_n^{c_1}, && \left[\hat{\partial}_n^{c_2}, \hat{x}^n\right] = 1 + \mathrm{i}ac_2\hat{\partial}_n^{c_2}, && \left[\hat{\partial}_n^{c_3}, \hat{x}^n\right] = 1 + 2\mathrm{i}a\hat{\partial}_n^{c_3}, \\
&\left[\hat{\partial}_n^{c_1}, \hat{x}^j\right] = 0, && \left[\hat{\partial}_n^{c_2}, \hat{x}^j\right] = \mathrm{i}a(1 + c_2)\hat{\partial}_j^{c_2}, && \left[\hat{\partial}_n^{c_3}, \hat{x}^j\right] = \mathrm{i}a\hat{\partial}_j^{c_3}, \\
&\left[\hat{\partial}_i^{c_1}, \hat{x}^n\right] = \mathrm{i}a\hat{\partial}_i^{c_1}, && \left[\hat{\partial}_i^{c_2}, \hat{x}^n\right] = 0, && \left[\hat{\partial}_i^{c_3}, \hat{x}^n\right] = \mathrm{i}ac_3\hat{\partial}_i^{c_3}, \\
&\left[\hat{\partial}_i^{c_1}, \hat{x}^j\right] = \delta_i^j, && \left[\hat{\partial}_i^{c_2}, \hat{x}^j\right] = \delta_i^j\left(1 + \mathrm{i}ac_2\hat{\partial}_n^{c_2}\right), && \left[\hat{\partial}_i^{c_3}, \hat{x}^j\right] = \delta_i^j.
\end{aligned} \tag{4}$$

The real parameters $c_i$ are not fixed by consistency with (2). We prefer to work with one particular choice in the following, $\hat{\partial}_\mu^{c_1=0}$. For brevity, $\hat{\partial}_\mu^{c_1=0}$ is denoted as $\hat{\partial}_\mu$. There is always more than one set of linear derivatives (consistent with the coordinate algebra) on noncommutative spaces of the Lie algebra type (1). If we denote the commutator of coordinates and derivatives linear in $\widehat{\partial}_\mu$ as

$$[\widehat{\partial}_\mu, \hat{x}^\nu] = \delta_\mu^\nu + \mathrm{i}\rho_\mu^{\nu\lambda}\widehat{\partial}_\lambda, \tag{5}$$

we obtain two conditions on $\rho_\mu^{\nu\lambda}$ from consistency with (1):

$$\rho_\lambda^{\mu\nu} - \rho_\lambda^{\nu\mu} = C_\lambda^{\mu\nu}, \qquad \rho_\lambda^{\mu\nu}\rho_\nu^{\kappa\sigma} - \rho_\lambda^{\kappa\nu}\rho_\nu^{\mu\sigma} = C_\nu^{\mu\kappa}\rho_\lambda^{\nu\sigma}. \tag{6}$$

All three one-parameter sets of derivatives $\hat{\partial}_\mu^{c_i}$ (4) fulfil the conditions (6). With the freedom indicated by the parametrization in (4), we have exhausted all linear derivatives. That there is such a variety of linear derivatives is disturbing at first sight. But all three families $\hat\partial^{c_i}_\mu$ can be mapped onto each other. The derivatives $\hat\partial_\mu$ ($c_1 = 0$) are mapped onto the derivatives $\hat\partial^{c_1}_\mu$ for arbitrary $c_1$ in the following way:

$$\hat\partial^{c_1}_j = \hat\partial_j, \qquad \hat\partial^{c_1}_n = \frac{\mathrm{e}^{\mathrm{i}ac_1\hat\partial_n} - 1}{\mathrm{i}ac_1}. \tag{7}$$

The role of shift operators is played by the following operators, in terms of $\hat\partial^{c_1}_\mu$

$$\mathrm{e}^{-\mathrm{i}a\hat\partial_n} = \left(\frac{1}{1 + \mathrm{i}c_1 a\hat\partial^{c_1}_n}\right)^{\frac{1}{c_1}}, \qquad \mathrm{e}^{\mathrm{i}a\hat\partial_n} = \left(1 + \mathrm{i}c_1 a\hat\partial^{c_1}_n\right)^{\frac{1}{c_1}}. \tag{8}$$

The derivatives $\hat\partial^{c_2}_\mu$ can be expressed in terms of $\hat\partial_\mu$ as well:

$$\hat\partial^{c_2}_n = \frac{\mathrm{e}^{\mathrm{i}ac_2\hat\partial_n} - 1}{\mathrm{i}ac_2}, \qquad \hat\partial^{c_2}_j = \hat\partial_j\, \mathrm{e}^{\mathrm{i}ac_2\hat\partial_n}, \qquad \mathrm{e}^{\mathrm{i}a\hat\partial_n} = \left(1 + \mathrm{i}c_2 a\hat\partial^{c_2}_n\right)^{\frac{1}{c_2}}. \tag{9}$$

The map from $\hat\partial_\mu$ to derivatives $\hat\partial^{c_3}_\mu$ reads

$$\hat\partial^{c_3}_n = \frac{\mathrm{e}^{2\mathrm{i}a\hat\partial_n} - 1}{2\mathrm{i}a} + \frac{\mathrm{i}ac_3}{2}\hat\partial_k\hat\partial_k, \qquad \hat\partial^{c_3}_j = \hat\partial_j, \qquad \mathrm{e}^{\mathrm{i}a\hat\partial_n} = \left(1 + 2\mathrm{i}a\hat\partial^{c_3}_n + a^2 c_3\hat\partial^{c_3}_l\hat\partial^{c_3}_l\right)^{\frac{1}{2}}. \tag{10}$$

The derivatives $\hat\partial_\mu$ are a very suitable basis in the algebra of derivatives to develop our formalism, as we will see later. The maps (7), (9) and (10) allow us to reformulate the entire formalism, which we will develop in the following in terms of $\hat\partial_\mu$, also in terms of any of the three one-parameter families of linear derivatives $\hat\partial^{c_i}_\mu$.

We will discuss several other derivatives in the following, for which we will lift the condition that the commutator (3) is linear in $\widehat{\partial_\mu}$.

### 2.2. *$SO_a(n)$ as a Hopf algebra*

We define the generators of rotations $M^{\mu\nu}$ by their commutation relations with the coordinates, demanding consistency with (2). We require undeformed transformation behaviour, i.e. the commutation relations of the Lie algebra of $SO(n)$, to zeroth order. In addition the generators of rotations $M^{rs}$ and $N^l = M^{nl}$ should appear at most linearly on the right-hand side of the commutators $[M^{\mu\nu}, \hat x^\rho]$. The only terms admissible in $\mathcal{O}(a)$ therefore involve the generators of rotations $M^{rs}$ and $N^l = M^{nl}$ exactly once. Higher-order terms in $a$ have to be accompanied by derivatives for dimensional reasons. If we demand[4] that the commutation relations close in coordinates and generators of rotations alone, the unique solution consistent with (2) is

$$\begin{aligned} &[M^{rs}, \hat x^n] = 0, && [N^l, \hat x^n] = \hat x^l + \mathrm{i}aN^l, \\ &[M^{rs}, \hat x^j] = \delta^{rj}\hat x^s - \delta^{sj}\hat x^r, && [N^l, \hat x^j] = -\delta^{lj}\hat x^n - \mathrm{i}aM^{lj}. \end{aligned} \tag{11}$$

The generators $M^{\mu\nu}$ have the commutation relations of the Lie algebra of $SO(n)$:

$$\begin{aligned} &[M^{rs}, M^{tu}] = \delta^{rt}M^{su} + \delta^{su}M^{rt} - \delta^{ru}M^{st} - \delta^{st}M^{ru}, \\ &[M^{rs}, N^i] = \delta^{ri}N^s - \delta^{si}N^r, \qquad [N^i, N^j] = M^{ij}. \end{aligned} \tag{12}$$

Even if, according to (12), the algebra of rotations is undeformed, the action on the coordinates is deformed (11). Therefore we will call $M^{\mu\nu}$ the generators of the algebra of $SO_a(n)$ rotations. Although actually the universal enveloping algebra of the Lie algebra of the symmetry group is deformed, we use the symbol of the symmetry group with a slight abuse of notation.

[4] More general commutation relations follow if this requirement is lifted.

We emphasise that consistency with the coordinate algebra leads to the so-called bicrossproduct basis of the $\kappa$-deformed Euclidean algebra, first defined in [7]. The bicrossproduct basis is singled out by (12) in contrast to the so-called classical basis which may be obtained by contracting the $q$-anti-de Sitter Hopf algebra $SO_q(3,2)$ [1]. The classical and the bicrossproduct basis are related by a nonlinear change of variables. We work in the bicrossproduct basis in the following. For all further constructions, consistency with (2) and (12) is a crucial requirement.

An important ingredient of the symmetry structure of $\kappa$-deformed space are the Leibniz rules of the generators of rotations and the derivatives[5]. They can be derived immediately from (4) and (11):

$$
\begin{aligned}
&\hat\partial_n\,(\hat f\cdot\hat g) = (\hat\partial_n\hat f)\cdot\hat g + \hat f\cdot(\hat\partial_n\hat g),\\
&\hat\partial_j(\hat f\cdot\hat g) = (\hat\partial_j\hat f)\cdot\hat g + (\mathrm{e}^{\mathrm{i}a\hat\partial_n}\hat f)\cdot(\hat\partial_j\hat g),\\
&M^{rs}(\hat f\cdot\hat g) = (M^{rs}\hat f)\cdot\hat g + \hat f\cdot(M^{rs}\hat g),\\
&N^l(\hat f\cdot\hat g) = (N^l\hat f)\cdot\hat g + (\mathrm{e}^{\mathrm{i}a\hat\partial_n}\hat f)\cdot(N^l\hat g) - \mathrm{i}a(\hat\partial_j\hat f)\cdot(M^{lj}\hat g)\;.
\end{aligned}
\tag{13}
$$

In a more technical language, equations (13) are the coproducts:

$$
\begin{aligned}
&\Delta\hat\partial_n = \hat\partial_n\otimes 1 + 1\otimes\hat\partial_n,\\
&\Delta\hat\partial_j = \hat\partial_j\otimes 1 + \mathrm{e}^{\mathrm{i}a\hat\partial_n}\otimes\hat\partial_j,\\
&\Delta M^{rs} = M^{rs}\otimes 1 + 1\otimes M^{rs},\\
&\Delta N^l = N^l\otimes 1 + \mathrm{e}^{\mathrm{i}a\hat\partial_n}\otimes N^l - \mathrm{i}a\hat\partial_j\otimes M^{lj}\;.
\end{aligned}
\tag{14}
$$

The notion of coproduct leads to the observation that the generators of the $\kappa$-deformed symmetry are elements of a Hopf algebra $\mathcal{A}$. A Hopf algebra is characterized by the specification of five operations on elements of a vector space: familiar operations are multiplication of vector space elements, the product ($\mathcal{V}\cdot\mathcal{W}\in\mathcal{A}$ if $\mathcal{V},\mathcal{W}\in\mathcal{A}$), and the unit $\mathbf{1}\in\mathcal{A}$ ($\mathcal{V}\cdot\mathbf{1}=\mathbf{1}\cdot\mathcal{V}=\mathcal{V}$). An *algebra* with a unit is a vector space which closes under multiplication of its elements.

The concept of *coalgebra* is in an abstract sense dual to the concept of an algebra. For a coalgebra two operations on vector space elements have to be specified: the coproduct $\Delta(\mathcal{V})$ and the counit $\epsilon(\mathcal{V})$. In the language of representations, the coproduct specifies how a coalgebra element $\mathcal{V}\in\mathcal{A}$ acts on products of representations. The counit describes the action on the zero-dimensional representation.

For a *bialgebra*, the algebra aspects and the coalgebra aspects have to be compatible. For a *Hopf algebra*, an additional operation, the antipode $S(\mathcal{V})$, has to be defined, compatible with all other operations. The antipode is the analogue of the inverse element of groups; in the language of representations, it states the action on the dual representation.

In our case we obtain

$$
\begin{aligned}
&\epsilon(\hat\partial_n)=0, && S(\hat\partial_n) = -\hat\partial_n,\\
&\epsilon(\hat\partial_j)=0, && S(\hat\partial_j) = -\hat\partial_j\,\mathrm{e}^{-\mathrm{i}a\hat\partial_n},\\
&\epsilon(M^{rs})=0, && S(M^{rs}) = -M^{rs},\\
&\epsilon(N^l)=0, && S(N^l) = -N^l\,\mathrm{e}^{-\mathrm{i}a\hat\partial_n} - \mathrm{i}aM^{lk}\hat\partial_k\,\mathrm{e}^{-\mathrm{i}a\hat\partial_n} - \mathrm{i}a(n-1)\hat\partial_l\,\mathrm{e}^{-\mathrm{i}a\hat\partial_n}.
\end{aligned}
\tag{15}
$$

We have introduced $M^{\mu\nu}$ in (11) as the generators of $SO_a(n)$ rotations. Since the coproduct involves derivatives, we can deform in a consistent way—as a Hopf algebra—only the (universal enveloping algebra of the) Lie algebra of the *inhomogeneous* $SO(n)$. Under

[5] We use the term 'Leibniz rule' also for the action of the generators of rotations on products of functions.

$SO_a(n)$ we will understand the deformed *Euclidean Hopf algebra*. For most of the following issues, considering $SO_a(n)$ as a bialgebra is sufficient.

For groups the inverse of the inverse is the identity and the dual representation of the dual representation is again the original one. Applying the antipode twice, we see that this is not necessarily the case for a deformed Hopf algebra such as $SO_a(n)$. We obtain $S^2(\mathcal{V}) = \mathcal{V}$ for $\mathcal{V} \neq N^i$ and for $N^i$

$$S^2(N^l) = N^l + \mathrm{i}a(n-1)\hat{\partial}_l \neq N^l. \tag{16}$$

In (4) derivatives $\hat{\partial}_\mu$ have been introduced as a minimal, linear deformation of commutative partial derivatives. These derivatives $\hat{\partial}_\mu$ $(c = 0)$ are a module of $SO_a(n)$; however, they have complicated commutation relations with the generators of rotations

$$\begin{aligned} &[M^{rs}, \hat{\partial}_n] = 0, \qquad &&[M^{rs}, \hat{\partial}_j] = \delta^r_j \hat{\partial}_s - \delta^s_j \hat{\partial}_r, \\ &[N^l, \hat{\partial}_n] = \hat{\partial}_l, \qquad &&[N^l, \hat{\partial}_j] = \delta^l_j \frac{1 - \mathrm{e}^{2\mathrm{i}a\hat{\partial}_n}}{2\mathrm{i}a} - \delta^l_j \frac{\mathrm{i}a}{2} \hat{\partial}_k \hat{\partial}_k + \mathrm{i}a \hat{\partial}_l \hat{\partial}_j. \end{aligned} \tag{17}$$

The other linear derivatives (4) are modules of $SO_a(n)$ as well.

We will demand, going beyond (12), that the algebra sector of the full deformed Euclidean Hopf algebra $SO_a(n)$ should remain undeformed. Equations (17) therefore force us to introduce other derivatives, which we will call Dirac derivatives, as generators of translations in $SO_a(n)$. This will be the content of the following subsection.

As an aside we note the representation of the orbital part of the generators of $SO_a(n)$ rotations $M^{rs}$ and $N^l$, in terms of $\hat{x}^\mu$ and $\hat{\partial}_\mu$:

$$\begin{aligned} \hat{M}^{rs} &= \hat{x}^s \hat{\partial}_r - \hat{x}^r \hat{\partial}_s, \\ \hat{N}^l &= \hat{x}^l \frac{\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n} - 1}{2\mathrm{i}a} - \hat{x}^n \hat{\partial}_l + \frac{\mathrm{i}a}{2} \hat{x}^l \hat{\partial}_k \hat{\partial}_k. \end{aligned} \tag{18}$$

This representation can be derived from (17) and it is consistent with (11) and (12).

### *2.3. Invariants and Dirac operator*

A familiar result [7] is that the lowest-order polynomial in the coordinates invariant under $SO_a(n)$ rotations is not $\hat{x}^\mu \hat{x}^\mu$ but

$$\hat{I}_1 = \hat{x}^\mu \hat{x}^\mu - \mathrm{i}a(n-1)\hat{x}^n. \tag{19}$$

This polynomial is not invariant in the sense $[M^{\mu\nu}, \hat{I}_1] = 0$, since

$$[M^{rs}, \hat{I}_1] = 0, \qquad [N^i, \hat{I}_1] = 2\mathrm{i}a\hat{x}^\mu M^{\mu i} + a^2(n-2)N^i. \tag{20}$$

The polynomial (19) can meaningfully be interpreted as an invariant, since another invariant (in the sense of (20)) is obtained if we multiply it with any $SO_a(n)$-invariant expression from the right.

Equation (19) is the lowest-order invariant in the coordinates alone. The Laplace operator $\hat{\Box}$ is the lowest order invariant built out of derivatives and it is truly invariant under $SO_a(n)$ rotations:

$$\hat{\Box} = \hat{\partial}_k \hat{\partial}_k \,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} + \frac{2}{a^2}(1 - \cos(a\hat{\partial}_n)), \qquad \text{with} \qquad [N^i, \hat{\Box}] = 0, \quad [M^{rs}, \hat{\Box}] = 0. \tag{21}$$

All functions $\hat{\Box} \cdot f(a^2\hat{\Box})$ of the Laplace operator are invariant and are consistent with the classical limit $\hat{\Box} = \hat{\partial}_\mu \hat{\partial}_\mu + \mathcal{O}(a)$.

The Dirac operator $\hat{D}$ is defined as the invariant under

$$[N^i, \hat{D}] + [n^i, \hat{D}] = 0, \qquad [M^{rs}, \hat{D}] + [m^{rs}, \hat{D}] = 0, \tag{22}$$

where $n^i = \frac{1}{4}[\gamma^n, \gamma^i]$ and $m^{rs} = \frac{1}{4}[\gamma^s, \gamma^r]$ are the generators of rotations for spinorial degrees of freedom, with the Euclidean $\gamma$-matrices $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$. The components of the Dirac operator $\hat{D} = \gamma^\mu \hat{D}_\mu$ will be called Dirac derivatives [15]. They are derivatives transforming linearly under $SO_a(n)$ rotations:

$$\begin{aligned} &[N^i, \hat{D}_n] = \hat{D}_i, && [N^i, \hat{D}_j] = -\delta^{ij}\hat{D}_n, \\ &[M^{rs}, \hat{D}_n] = 0, && [M^{rs}, \hat{D}_j] = \delta^r_j \hat{D}_s - \delta^s_j \hat{D}_r. \end{aligned} \tag{23}$$

There is a continuous range of solutions to (23) with classical limit $\hat{D}_\mu = \hat{\partial}_\mu + \mathcal{O}(a)$:

$$\begin{aligned} \hat{D}_n &= \left(\frac{1}{a}\sin(a\hat{\partial}_n) + \frac{\mathrm{i}a}{2}\hat{\partial}_k\hat{\partial}_k \,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\right) f(a^2\hat{\Box}), \\ \hat{D}_j &= \hat{\partial}_j \,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} f(a^2\hat{\Box}). \end{aligned} \tag{24}$$

The simplest solution of (23) is the one with $f = 1$. We choose this solution to be *the* Dirac derivative. It is a nonlinear derivative in the sense of (3):

$$\begin{aligned} &[\hat{D}_n, \hat{x}^i] = \mathrm{i}a\hat{D}_i, \\ &[\hat{D}_n, \hat{x}^n] = \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu} = 1 - \frac{a^2}{2}\hat{\Box}, \\ &[\hat{D}_j, \hat{x}^i] = \delta^i_j\left(-\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right) = \delta^i_j\left(1 - \mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}\right), \\ &[\hat{D}_j, \hat{x}^n] = 0. \end{aligned} \tag{25}$$

Its coproduct is given by

$$\begin{aligned} \Delta\hat{D}_n &= \hat{D}_n \otimes \left(-\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right) + \frac{\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k} \otimes \hat{D}_n \\ &\quad + \mathrm{i}a\hat{D}_i \frac{\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k} \otimes \hat{D}_i, \\ \Delta\hat{D}_j &= \hat{D}_j \otimes \left(-\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}\right) + 1 \otimes \hat{D}_j. \end{aligned} \tag{26}$$

The Dirac derivative together with the generators of $SO_a(n)$ rotations $M^{\mu\nu}$ forms a $\kappa$-deformed Euclidean Hopf algebra which is undeformed in the algebra sector, (12) and (23). The deformation is purely in the coalgebra sector, (14) and (26). This special basis of $SO_a(n)$ we will refer to in the following as *the* $SO_a(n)$. Recall that it is not unique (24). Together with the counit and the antipode of the Dirac derivative

$$\begin{aligned} &\epsilon(\hat{D}_n) = 0, && S(\hat{D}_n) = -\hat{D}_n + \mathrm{i}a\hat{D}_l\hat{D}_l \frac{\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k}, \\ &\epsilon(\hat{D}_j) = 0, && S(\hat{D}_j) = -\hat{D}_j \frac{\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{1 - a^2\hat{D}_k\hat{D}_k}, \end{aligned} \tag{27}$$

and the property $S^2(\hat{D}_\mu) = \hat{D}_\mu$, all operations of the full Euclidean Hopf algebra $SO_a(n)$ have been defined.

The square of the Dirac derivative is not identical to the Laplace operator, but $\hat{D}_\mu\hat{D}_\mu = \hat{\Box}\left(1 - \frac{a^2}{4}\hat{\Box}\right)$. However, having in mind the caveats below equations (21) and (24) we could rescale the Dirac derivative $\hat{D}'_\mu = \frac{\hat{D}_\mu}{\sqrt{1-\frac{a^2}{4}\hat{\Box}}}$ such that $\hat{D}'_\mu \hat{D}'_\mu = \hat{\Box}$. We do not follow this train of thought in this paper.

We also quote the antipode $S(\hat{\Box}) = \hat{\Box}$ of the Laplace operator, its commutators with coordinates $[\hat{\Box}, \hat{x}^\mu] = 2\hat{D}_\mu$ and its Leibniz rule

$$\begin{aligned}\hat{\Box}\,(\hat{f}\cdot\hat{g}) &= (\hat{\Box}\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{\Box}\hat{g}) \\ &\quad + 2(\hat{D}_i\,\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\hat{D}_i\hat{g}) + \frac{2}{a^2}((1-\mathrm{e}^{\mathrm{i}a\hat{\partial}_n})\hat{f})\cdot((1-\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n})\hat{g}). \end{aligned} \tag{28}$$

In (28) we have used the identities:

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} &= -\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu} = 1 - \mathrm{i}a\hat{D}_n - \frac{a^2}{2}\hat{\Box}, \\ \mathrm{e}^{\mathrm{i}a\hat{\partial}_n} &= \frac{\mathrm{i}a\hat{D}_n + \sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_j\hat{D}_j}. \end{aligned} \tag{29}$$

There are also further invariants, such as in four-dimensional $\kappa$-Minkowski spacetime the Pauli–Lubanski vector, which has been discussed in [2] and in [16] in the bicrossproduct basis. From (12) and (23) the generalization of the Pauli–Lubanski vector in $n = 2m$ Euclidean dimensions can be deduced:

$$\begin{aligned} &W_{i+1}^2 = W_{\mu_1\cdots\mu_{2i-1}}W_{\mu_1\cdots\mu_{2i-1}}, \qquad W_1^2 = \hat{D}_\mu\hat{D}_\mu, \qquad i = 1,\ldots,\frac{n-2}{2}, \\ &W_{\mu_1\cdots\mu_{2i-1}} = \epsilon_{\mu_1\cdots\mu_n}M^{\mu_{2i}\mu_{2i+1}}\cdots M^{\mu_{n-2}\mu_{n-1}}\hat{D}_{\mu_n}. \end{aligned} \tag{30}$$

All invariants should be identical to their undeformed counterparts, exchanging ordinary derivatives with the Dirac derivative. The reason is that the algebra sector in our particular basis of $SO_a(n)$ is undeformed.

## 3. Star product and operator representations

### *3.1. Different ⋆-products*

In this section we represent the associative algebra of functions of noncommuting coordinates as an algebra of functions of commuting variables by means of ⋆-products. This allows a representation of the generators $\hat{D}_\mu$ and $M^{\mu\nu}$ of the Hopf algebra $SO_a(n)$ in terms of differential operators of ordinary, commuting derivatives and coordinates. The representation by means of ⋆-products is particularly suitable, since it allows an expansion order by order in $a$.

The ⋆-product replaces the point-wise product of commutative spacetime. For a given noncommutative associative algebra of coordinates, there are generically several different ⋆-products fulfilling

$$[x^\mu \overset{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = \mathrm{i}C^{\mu\nu}_\lambda x^\lambda, \quad \text{if} \quad [\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}C^{\mu\nu}_\lambda \hat{x}^\lambda \tag{31}$$

in the Lie algebra case for example. Many interesting ⋆-products simply reproduce different ordering prescriptions imposed on the abstract algebra of coordinates. For noncommutative coordinates the order in a monomial has to be specified, otherwise one would miscount the elements of the basis of monomials. After multiplying two functions of noncommutative variables, the product has to be reordered according to the chosen prescription. The ⋆-products of interest here perform this reordering for commuting coordinates $x^\mu$.

In the case of $\kappa$-deformed space with only one noncommuting coordinate $\hat{x}^n$, three ordering prescriptions can be chosen in a generic way: all factors of $\hat{x}^n$ to the far left, all factors of $\hat{x}^n$ to the far right or complete symmetrization of all coordinates. We are especially interested in the symmetric $\star$-product because of its Hermiticity:

$$\overline{f(x) \star g(x)} = \overline{g}(x) \star \overline{f}(x). \tag{32}$$

Note that we set $f(x) \star g(x) \equiv (f \star g)(x)$, in other words we omit the multiplication map of the usual definition of the $\star$-product. This abuse of the usual notation allows us to simplify the coproduct formulae in the following.

An important condition on a $\star$-product is that it is associative. For Lie algebra models such as $\kappa$-deformed space with symmetric ordering, the Baker–Campbell–Hausdorff (BCH) formula can be used to obtain an associative symmetric $\star$-product. The BCH expansion involves the structure constants of the Lie algebra (1):

$$\begin{aligned} f(x) \star g(x) = \exp\Bigg(&\frac{\mathrm{i}}{2} x^\lambda C_\lambda^{\mu\nu} \partial_\mu \otimes \partial_\nu + \frac{1}{12} x^\lambda C_\lambda^{\rho\sigma} C_\rho^{\mu\nu} (\partial_\sigma \otimes 1 - 1 \otimes \partial_\sigma) \partial_\mu \otimes \partial_\nu \\ &+ \frac{\mathrm{i}}{24} x^\lambda C_\lambda^{\alpha\beta} C_\alpha^{\rho\sigma} C_\rho^{\mu\nu} \partial_\beta \partial_\mu \otimes \partial_\sigma \partial_\nu + \cdots \Bigg) \; f(y) \otimes g(z)|_{y,z\to x}. \end{aligned} \tag{33}$$

Generically there is no closed symbolical form, such as the Moyal–Weyl product, for $\star$-products of Lie algebra noncommutative spaces. For the $\kappa$-deformed space, however, there exists a closed formula, as we will show in the following. This result has been found before [17, 18]. For our derivation we use the notion of equivalent $\star$-products. Two $\star$-products $\star$ and $\star'$ are equivalent if they can be related by a differential operator $T$:

$$T(f(x) \star g(x)) = T(f(x)) \star' T(g(x)). \tag{34}$$

For example, $\star$-products which represent the same algebra with a different ordering prescription are equivalent. We use this fact to relate the symmetric $\star$-product to the normal ordered $\star$-products. These in turn can be derived via a Weyl quantization procedure (cf [11]):

$$\begin{aligned} f(x) \star_L g(x) &= \lim_{\substack{y\to x\\ z\to x}} \exp\left(x^j \frac{\partial}{\partial y^j}\left(\mathrm{e}^{-\mathrm{i}a\frac{\partial}{\partial z^n}} - 1\right)\right) f(y) g(z), \\ f(x) \star_R g(x) &= \lim_{\substack{y\to x\\ z\to x}} \exp\left(x^j \frac{\partial}{\partial z^j}\left(\mathrm{e}^{\mathrm{i}a\frac{\partial}{\partial y^n}} - 1\right)\right) f(y) g(z). \end{aligned} \tag{35}$$

The $\star$-product $\star_L$ reproduces an ordering for which all $\hat{x}^n$ stand on the left-hand side in any monomial; $\star_R$ reproduces the opposite ordering prescription. For our derivation we need several formulae, which follow from (35) and from properties of the BCH formula (cf [19]):

$$\begin{aligned} &x^j \star_L f(x) = x^j\, \mathrm{e}^{-\mathrm{i}a\partial_n} f(x), && f(x) \star_L x^j = x^j f(x), \\ &x^n \star_L f(x) = x^n f(x), && f(x) \star_L x^n = (x^n - \mathrm{i}a x^k \partial_k) f(x), \\ &x^j \star_R f(x) = x^j f(x), && f(x) \star_R x^j = x^j\, \mathrm{e}^{\mathrm{i}a\partial_n} f(x), \\ &x^n \star_R f(x) = (x^n + \mathrm{i}a x^k \partial_k) f(x), && f(x) \star_R x^n = x^n f(x). \\ &x^j \star f(x) = x^j \frac{\mathrm{i}a\partial_n}{\mathrm{e}^{\mathrm{i}a\partial_n} - 1} f(x), && f(x) \star x^j = x^j \frac{-\mathrm{i}a\partial_n}{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1} f(x), \\ &x^n \star f(x) = \left(x^n - \frac{x^k \partial_k}{\partial_n}\left(\frac{\mathrm{i}a\partial_n}{\mathrm{e}^{\mathrm{i}a\partial_n} - 1} - 1\right)\right) f(x), \\ &f(x) \star x^n = \left(x^n - \frac{x^k \partial_k}{\partial_n}\left(\frac{-\mathrm{i}a\partial_n}{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1} - 1\right)\right) f(x). \end{aligned} \tag{36}$$

The operator $T$ up to first order in $a$ has to be of the form $T = 1 + \mathrm{i}acx^j\partial_j\partial_n + \cdots$, with a real constant $c$ to be determined. Note that $T$ also depends on the coordinates $x^j$. We obtain $T(x^\mu) = x^\mu$ and for the left ordered $\star$-product $\star_L$

$$T(x^j \star_L g(x)) = T(x^j) \star T(g(x)) \quad \Rightarrow \quad T(x^j \mathrm{e}^{\mathrm{i}a\partial_n} f(x)) = x^j \frac{\mathrm{i}a\partial_n}{\mathrm{e}^{\mathrm{i}a\partial_n} - 1} T(f(x)). \tag{37}$$

Formula (37) can be written as a differential equation with the unique solution

$$T = \lim_{z\to x} \exp\left(z^j\partial_{x^j}\left(\frac{-\mathrm{i}a\partial_n}{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1} - 1\right)\right), \qquad T^{-1} = \lim_{z\to x} \exp\left(z^j\partial_{x^j}\left(\frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n} - 1\right)\right), \tag{38}$$

demanding that $T \cdot T^{-1} = 1$.

Similar equivalence operators $\tilde{T}$ can be constructed relating $\star$ and $\star_R$, $\tilde{T}(f \star_R g) = \tilde{T}(f) \star \tilde{T}(g)$. The result is $\tilde{T} = \lim_{z\to x} \exp\left(z^j\partial_{x^j}\left(\frac{\mathrm{i}a\partial_n}{\mathrm{e}^{\mathrm{i}a\partial_n}-1} - 1\right)\right)$.

With this solution for $T$ we construct the symmetric $\star$-product:

$$\begin{aligned}
f(x) \star g(x) &= \lim_{\substack{y\to x\\ z\to x}} T(T^{-1}(f(z)) \star_L T^{-1}(g(y))) \\
&= \lim_{w\to x} \exp\left(x^j\partial_{w^j}\left(\frac{-\mathrm{i}a\partial_{w^n}}{\mathrm{e}^{-\mathrm{i}a\partial_{w^n}} - 1} - 1\right)\right) \lim_{\substack{z\to w\\ y\to w}} \exp(w^j\partial_{z^j}(\mathrm{e}^{-\mathrm{i}a\partial_{y^n}} - 1)) \\
&\quad \times \lim_{\substack{u\to z\\ t\to y}} \left(\exp\left(z^j\partial_{u^j}\left(\frac{\mathrm{e}^{-\mathrm{i}a\partial_{u^n}} - 1}{-\mathrm{i}a\partial_{u^n}} - 1\right)\right) f(u)\right) \\
&\quad \times \left(\exp\left(y^j\partial_{t^j}\left(\frac{\mathrm{e}^{-\mathrm{i}a\partial_{t^n}} - 1}{-\mathrm{i}a\partial_{t^n}} - 1\right)\right) g(t)\right).
\end{aligned} \tag{39}$$

Contracting all limits, this result is written in a compact way ($\partial_n = \partial_{y^n} + \partial_{z^n}$):

$$\begin{aligned}
f(x) \star g(x) = \lim_{\substack{y\to x\\ z\to x}} \exp\Bigg( & x^j\partial_{y^j}\left(\frac{\partial_n}{\partial_{y^n}}\,\mathrm{e}^{-\mathrm{i}a\partial_{z^n}}\,\frac{1 - \mathrm{e}^{-\mathrm{i}a\partial_{y^n}}}{1 - \mathrm{e}^{-\mathrm{i}a\partial_n}} - 1\right) \\
& + x^j\partial_{z^j}\left(\frac{\partial_n}{\partial_{z^n}}\,\frac{1 - \mathrm{e}^{-\mathrm{i}a\partial_{z^n}}}{1 - \mathrm{e}^{-\mathrm{i}a\partial_n}} - 1\right)\Bigg) f(y)g(z).
\end{aligned} \tag{40}$$

### *3.2. Representation of derivatives and generators of rotations*

The symmetry algebra of $\kappa$-deformed space, i.e. the generators of rotations and the Dirac derivatives, can be represented as differential operators on spaces of commuting variables with the symmetric $\star$-product as multiplication:

$$\hat{\partial}_n \hat{f} \longrightarrow \partial_n^* f(x) = \partial_n f(x),$$

$$\hat{\partial}_j \hat{f} \longrightarrow \partial_j^* f(x) = \partial_j\left(\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{\mathrm{i}a\partial_n}\right) f(x),$$

$$N^l \hat{f} \longrightarrow N^{*l} f(x) = \left(x^l\partial_n - x^n\partial_l + x^l\partial_\mu\partial_\mu\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1}{2\partial_n} - x^\mu\partial_\mu\partial_l\frac{\mathrm{e}^{\mathrm{i}a\partial_n} - 1 - \mathrm{i}a\partial_n}{\mathrm{i}a\partial_n^2}\right) f(x),$$

$$M^{rs}\hat{f} \longrightarrow M^{*rs} f(x) = (x^s\partial_r - x^r\partial_s) f(x),$$

$$\hat{D}_n \hat{f} \longrightarrow D_n^* f(x) = \left(\frac{1}{a}\sin(a\partial_n) - \frac{1}{\mathrm{i}a\partial_n\partial_n}\partial_k\partial_k(\cos(a\partial_n) - 1)\right) f(x),$$

$$\hat{D}_j \hat{f} \longrightarrow D_j^* f(x) = \partial_j \left( \frac{\mathrm{e}^{-\mathrm{i}a\partial_n} - 1}{-\mathrm{i}a\partial_n} \right) f(x),$$

$$\hat{\Box} \hat{f} \longrightarrow \Box^* f(x) = \partial_\mu \partial_\mu \; \frac{2(1 - \cos(a\partial_n))}{a^2 \partial_n \partial_n} f(x). \tag{41}$$

This representation can be derived in a perturbation expansion on symmetrized monomials multiplied with the $\star$-product. However, it is easier to derive it using the expressions $x^\mu \star f(x)$ in (36). Rewriting these expressions symbolically as $x^{*\mu} f(x)$, relations such as

$$[\partial_j^*, x^{*n}] f(x) = \partial_j^* x^{*n} f(x) - x^{*n} \partial_j^* f(x) \overset{!}{=} \mathrm{i}a \partial_j^* f(x) \tag{42}$$

have to be fulfilled for arbitrary $f(x)$. Note that all operators in (41) coincide to zeroth order in $a$ with their commutative counterparts.

Similar representations can be derived for the normal ordered $\star$-products. These representations have to be different from (41), since the same abstract algebra of $\hat{\partial}_\mu$, $\hat{D}_\mu$, $M^{rs}$ and $N^l$ is represented on different, but equivalent $\star$-products. The different $\star$-representations can be related with equivalence operators, via $\partial_j^{*_L} = T^{-1} \partial_j^* T$ etc.

For the left ordered $\star$-product ($\star_L$) we obtain

$$\begin{aligned}
\partial_n^{*_L} f(x) &= \partial_n f(x), \\
\partial_j^{*_L} f(x) &= \partial_j \, \mathrm{e}^{\mathrm{i}a\partial_n} f(x), \\
N^{*_L\, l} f(x) &= \left( x^l \frac{1}{a} \sin(a\partial_n) - x^n \partial_l \, \mathrm{e}^{\mathrm{i}a\partial_n} + \frac{\mathrm{i}a}{2} x^l \partial_k \partial_k \, \mathrm{e}^{\mathrm{i}a\partial_n} \right) f(x), \\
M^{*_L\, rs} f(x) &= (x^s \partial_r - x^r \partial_s) f(x), \\
D_n^{*_L} f(x) &= \left( \frac{1}{a} \sin(a\partial_n) + \frac{\mathrm{i}a}{2} \partial_k \partial_k \, \mathrm{e}^{\mathrm{i}a\partial_n} \right) f(x), \\
D_j^{*_L} f(x) &= \partial_j f(x), \\
\Box^{*_L} f(x) &= \left( -\frac{2}{a^2} (\cos(a\partial_n) - 1) + \partial_k \partial_k \, \mathrm{e}^{\mathrm{i}a\partial_n} \right) f(x).
\end{aligned} \tag{43}$$

The result for the right ordered $\star$-product ($\star_R$) is

$$\begin{aligned}
\partial_n^{*_R} f(x) &= \partial_n f(x), \\
\partial_j^{*_R} f(x) &= \partial_j f(x), \\
N^{*_R l} f(x) &= \left( x^l \frac{1}{2\,\mathrm{i}a} (\mathrm{e}^{2\mathrm{i}a\partial_n} - 1) - x^n \partial_l - \mathrm{i}a x^k \partial_k \partial_l + \frac{\mathrm{i}a}{2} x^l \partial_k \partial_k \right) f(x), \\
M^{*_R\, rs} f(x) &= (x^s \partial_r - x^r \partial_s) f(x), \\
D_n^{*_R} f(x) &= \left( \frac{1}{a} \sin(a\partial_n) + \frac{\mathrm{i}a}{2} \partial_k \partial_k \, \mathrm{e}^{-\mathrm{i}a\partial_n} \right) f(x), \\
D_j^{*_R} f(x) &= \partial_j \, \mathrm{e}^{-\mathrm{i}a\partial_n} f(x), \\
\Box^{*_R} f(x) &= \left( -\frac{2}{a^2} (\cos(a\partial_n) - 1) + \partial_k \partial_k \, \mathrm{e}^{-\mathrm{i}a\partial_n} \right) f(x).
\end{aligned} \tag{44}$$

Summarizing the ambiguities of our construction: we have made the choice that the $SO_a(n)$ Hopf algebra is the one with undeformed algebra sector. Equation (11) followed uniquely from (2). In contrast, we have made an arbitrary choice following (24), choosing the simplest Dirac derivative out of a continuous range of solutions. The only other choice is the $\star$-product to represent the abstract algebra. Hermiticity is a strong argument for favouring the symmetric $\star$-product.

## 4. Forms

### *4.1. Vector-like transforming one-forms*

A crucial ingredient of a geometric approach towards the $\kappa$-deformed space is the exterior differential, denoted by d. In order to find a representation of d, a working definition of a one-form is needed.

The expected properties of d are

- nilpotency: $\mathrm{d}^2 = 0$;
- application of d to a coordinate gives a one-form $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$;
- invariance under $SO_a(n)$: $[M^{rs}, \mathrm{d}] = 0$, $[N^l, \mathrm{d}] = 0$;
- undeformed Leibniz rule: $\mathrm{d}(\hat{f} \cdot \hat{g}) = (\mathrm{d}\hat{f}) \cdot \hat{g} + \hat{f} \cdot \mathrm{d}\hat{g}$.

Demanding invariance of d under $SO_a(n)$, a natural ansatz is that the Dirac derivative $\hat{D}_\mu$ is the convenient derivative dual to a set of vector-like transforming one-forms $\hat{\xi}^\mu$, $\mathrm{d} = \hat{\xi}^\mu \hat{D}_\mu$:

$$[M^{rs}, \hat{\xi}^\mu] = \delta^{r\mu}\hat{\xi}^s - \delta^{s\mu}\hat{\xi}^r, \qquad [N^l, \hat{\xi}^\mu] = \delta^{n\mu}\hat{\xi}^l - \delta^{l\mu}\hat{\xi}^n. \tag{45}$$

The nilpotency of $\mathrm{d}^2 = 0$ can be achieved requiring that one-forms $\hat{\xi}^\mu$ commute with derivatives and anti-commute among themselves $\{\hat{\xi}^\mu, \hat{\xi}^\nu\} = 0$.

Demanding that the commutator of d with a coordinate is a one-form, $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$ is a sufficient condition for an undeformed Leibniz rule of d.

If we add the condition that the commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ close in the space of one-forms alone, there is no differential calculus consisting of $n$ one-forms fulfilling all these conditions simultaneously. Under this additional condition, a familiar result (e.g. [14]) states that the basis of one-forms is $(n + 1)$-dimensional.

There have been hints towards this result in our discussion of the Dirac operator (25). Its commutator with the coordinates $[\hat{D}_\mu, \hat{x}^\nu]$ is a power series in the Dirac derivative alone, but it is linear when adding the Laplace operator $\hat{\Box}$. The $(n + 1)$-dimensional set of derivatives $(\hat{D}_\mu, \hat{\Box})$ is dual to the $(n + 1)$-dimensional set of one-forms $(\widehat{\mathrm{d}x}^\mu, \hat{\phi})$ introduced in [14]

$$\begin{aligned}
&\mathrm{d} = \widehat{\mathrm{d}x}^n \hat{D}_n + \widehat{\mathrm{d}x}^j \hat{D}_j - \frac{a^2}{2}\hat{\phi}\,\hat{\Box}, && [\mathrm{d}, \hat{x}^\mu] = \widehat{\mathrm{d}x}^\mu, \\
&[\widehat{\mathrm{d}x}^n, \hat{x}^n] = a^2\hat{\phi}, \quad [\widehat{\mathrm{d}x}^j, \hat{x}^n] = 0, && [\hat{\phi}, \hat{x}^n] = -\widehat{\mathrm{d}x}^n, \\
&[\widehat{\mathrm{d}x}^n, \hat{x}^i] = \mathrm{i}a\widehat{\mathrm{d}x}^i, \quad [\widehat{\mathrm{d}x}^j, \hat{x}^i] = -\mathrm{i}a\delta^{ij}\widehat{\mathrm{d}x}^n + a^2\,\delta^{ij}\hat{\phi}, && [\hat{\phi}, \hat{x}^i] = -\widehat{\mathrm{d}x}^i.
\end{aligned} \tag{46}$$

It is a frequent observation in noncommutative geometry that the set of one-forms on a particular space has one element more than in the commutative setting. In our case this is acceptable at first sight since for $a \to 0$, $\mathrm{d} \to \mathrm{d}_{\mathrm{class}}$. But several problems remain. Only $n$ one-forms can be obtained by applying d to the coordinates. A gauge theory with gauge potentials as one-forms would result in an additional degree of freedom in the gauge potentials. The cohomology of the differential calculus has an entirely different structure than in the commutative case.

We will therefore follow a different strategy and demand as a central condition that there are only $n$ one-forms on the noncommutative space. Of course we will not get this condition for free; there will be a trade-off of the kind that the one-forms $\hat{\xi}^\mu$ will have derivative-valued commutation relations with the coordinates.

To calculate the one-forms, we start from their commutators with the coordinates

$$\hat{\xi}^\nu \overset{!}{=} [\mathrm{d}, \hat{x}^\nu] = [\hat{\xi}^\mu \hat{D}_\mu, \hat{x}^\nu] = [\hat{\xi}^\mu, \hat{x}^\nu]\hat{D}_\mu + \mathrm{i}a\hat{\xi}^n \hat{D}_\nu + \hat{\xi}^\nu(-\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu \hat{D}_\mu})). \tag{47}$$

From (47) follows that the commutator $[\hat{\xi}^\mu, \hat{x}^\nu]$ will involve derivatives. To calculate it we have made a general ansatz with *derivative-valued* commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ involving all terms compatible with the index structure. The result is derived requiring that the commutators $[\hat{\xi}^\mu, \hat{x}^\nu]$ are compatible with (2). Invariance under $SO_a(n)$ rotations does not add further constraints and we find the unique solution

$$[\hat{\xi}^\mu, \hat{x}^\nu] = \mathrm{i}a(\delta^{\mu n}\hat{\xi}^\nu - \delta^{\mu\nu}\hat{\xi}^n) + (\hat{\xi}^\mu \hat{D}_\nu + \hat{\xi}^\nu \hat{D}_\mu - \delta^{\mu\nu}\hat{\xi}^\rho \hat{D}_\rho)\frac{1 - \sqrt{1 - a^2\hat{D}_\sigma \hat{D}_\sigma}}{\hat{D}_\lambda \hat{D}_\lambda}. \tag{48}$$

As an aside note that $\frac{1-\sqrt{1-a^2\hat{D}_\sigma \hat{D}_\sigma}}{\hat{D}_\lambda \hat{D}_\lambda} = \frac{a^2}{2}\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$.

The price we have to pay for having only $n$ one-forms is that the commutator (48) is highly nonlinear in the Dirac derivative.

It is possible to generalize one of the conditions for the differential calculus, the undeformed Leibniz rule $[\mathrm{d}, \hat{x}^\mu] = \hat{\xi}^\mu$. We define commutation relations between a second set of one-forms $\widetilde{\xi}^\mu$ and coordinates $\hat{x}^\nu$, consistent with (2) and (12). For these one-forms $\widetilde{\xi}^\mu$ the application of d to a coordinate does not return the one-form, but a derivative-valued expression

$$[\widetilde{\xi}^n \hat{D}_n + \widetilde{\xi}^j \hat{D}_j, \hat{x}^\nu] = [\mathrm{d}, \hat{x}^\nu] = (\mathrm{d}\hat{x})^\nu = \widetilde{\xi}^\nu \cdot f(\hat{D}_n, \hat{D}_j\hat{D}_j), \tag{49}$$

with a suitable function of the Dirac derivative $f(\hat{D}_n, \hat{D}_j\hat{D}_j)$.

The most general solution for (49) is

$$\begin{aligned}
[\mathrm{d}, \hat{x}^\nu] &= \widetilde{\xi}^\nu + c'\widetilde{\xi}^\nu\left(\frac{1}{\sqrt{1 - a^2\hat{D}_\sigma \hat{D}_\sigma}} - 1\right) \\
[\widetilde{\xi}^\mu, \hat{x}^\nu] &= \mathrm{i}a(\delta^{\mu n}\widetilde{\xi}^\nu - \delta^{\mu\nu}\widetilde{\xi}^n) + (1 - c')(\widetilde{\xi}^\mu \hat{D}_\nu + \widetilde{\xi}^\nu \hat{D}_\mu - \delta^{\mu\nu}\widetilde{\xi}^\rho \hat{D}_\rho)\frac{1 - \sqrt{1 - a^2\hat{D}_\sigma \hat{D}_\sigma}}{\hat{D}_\lambda \hat{D}_\lambda} \\
&\quad + c'\,\widetilde{\xi}^\nu \hat{D}_\mu \frac{a^2}{\sqrt{1 - a^2\hat{D}_\lambda \hat{D}_\lambda}},
\end{aligned} \tag{50}$$

for an arbitrary constant $c'$. The solution (48) corresponds to $c' = 0$ and we will use it exclusively in the following.

With (48) it is very difficult to calculate the action of d on a general $x$-dependent one-form $\alpha_\mu(\hat{x})\hat{\xi}^\mu$:

$$\mathrm{d}\alpha = \mathrm{d}(\alpha_\mu(\hat{x})\hat{\xi}^\mu) = \hat{\xi}^\nu(\hat{D}_\nu\alpha_\mu(\hat{x}))\hat{\xi}^\mu \neq (\hat{D}_\nu\alpha_\mu(\hat{x}))\hat{\xi}^\nu\hat{\xi}^\mu. \tag{51}$$

However, a general one-form may be defined in such a way that $\hat{\xi}^\mu$ stands to the left of the coefficient function:

$$\mathrm{d}\alpha = \mathrm{d}(\hat{\xi}^\mu\alpha_\mu(\hat{x})) = \hat{\xi}^\nu\hat{\xi}^\mu(\hat{D}_\nu\alpha_\mu(\hat{x})). \tag{52}$$

Still it is interesting to see whether there are one-forms which allow an action of d as in (51), independent of the order. This motivates the introduction of a second basis of one-forms, which we call the frame.

### 4.2. Representation of $\hat{\xi}^\mu$

In analogy with the approach in section 3.2 we now derive a representation of the one-forms $\hat{\xi}^\mu \to \xi^{*\mu}$ in the $\star$-product setting.

The starting point is the commutator (48), which involves a power series expansion in the derivatives. This implies that the $\xi^{*\mu}$ can be written as functions of the commutative one-forms $\mathrm{d}x^\mu$ and the commutative derivatives $\partial_\nu$. The one-forms $\xi^{*\mu}$ should be at most linear in $\mathrm{d}x^\mu$.

We make the most general ansatz compatible with the index structure

$$\begin{aligned}\xi^{*n} &= \mathrm{d}x^n e_1(\partial_i\partial_i, \partial_n) + \mathrm{d}x^k \partial_k e_2(\partial_i\partial_i, \partial_n),\\ \xi^{*j} &= \mathrm{d}x^j f_1(\partial_i\partial_i, \partial_n) + \mathrm{d}x^n \partial_j f_2(\partial_i\partial_i, \partial_n) + \mathrm{d}x^k \partial_k\partial_j f_3(\partial_i\partial_i, \partial_n).\end{aligned} \tag{53}$$

We have stated in (42) how to calculate the representation of a derivative operator from its commutator with a coordinate. We collect the terms proportional to different one-forms $\mathrm{d}x^\mu$ and different combinations of derivatives. We obtain an overdetermined system of equations which can be solved consistently. With the abbreviation

$$\gamma = \frac{1}{1 + \frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)}, \tag{54}$$

we obtain the result

$$\begin{aligned} f_1 &= \gamma, & e_1 &= \left(1 + \cos(a\partial_n) - \frac{\partial_k\partial_k}{\partial_n^2}(\cos(a\partial_n) - 1)\right)\gamma^2,\\ f_2 &= -\frac{2\mathrm{i}}{\partial_n}\sin(a\partial_n)\gamma^2, & e_2 &= \frac{2\mathrm{i}}{\partial_n}\sin(a\partial_n)\gamma^2,\\ f_3 &= -\frac{1}{\partial_n^2}(\cos(a\partial_n) - 1)\gamma^2, \end{aligned} \tag{55}$$

or

$$\begin{aligned}\xi^{*n} &= \left(\mathrm{d}x^n(1 + \cos(a\partial_n) - \frac{\partial_k\partial_k}{\partial_n^2}(\cos(a\partial_n) - 1) + \mathrm{d}x^k\frac{2\mathrm{i}\partial_k}{\partial_n}\sin(a\partial_n)\right)\gamma^2,\\ \xi^{*j} &= \left(\mathrm{d}x^j\left(1 + \frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)\right) - \mathrm{d}x^n\frac{2\mathrm{i}\partial_j}{\partial_n}\sin(a\partial_n)\right.\\ &\qquad \left. - \mathrm{d}x^k\frac{2\partial_k\partial_j}{\partial_n^2}(\cos(a\partial_n) - 1)\right)\gamma^2.\end{aligned} \tag{56}$$

The more general differential calculus (50) has a particularly simple solution for $c' = 1$. The one-forms $\widetilde{\xi}^\mu$ for $c' = 1$ have the following $\star$-representation $\widetilde{\xi}^{*\mu}$:

$$\begin{aligned}\widetilde{\xi}^{*n} &= \left(\mathrm{d}x^n + \mathrm{d}x^l\frac{\partial_l}{\partial_n}(1 - \mathrm{e}^{-\mathrm{i}a\partial_n})\right)\frac{1}{1 + \frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)},\\ \widetilde{\xi}^{*j} &= \left(-\mathrm{d}x^l\frac{\partial_l\partial_j}{\partial_n\partial_n}(\cos(a\partial_n) - 1) + \mathrm{d}x^n\frac{\partial_j}{\partial_n}(1 - \mathrm{e}^{-\mathrm{i}a\partial_n})\right)\frac{1}{1 + \frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)}.\end{aligned} \tag{57}$$

It is interesting to note that for this specific set of differentials $\widetilde{\xi}^{*\mu}$ we obtain a representation on the space of functions multiplied with the $\star$-product, where $\widetilde{\xi}^{*j}$ is not proportional to $\mathrm{d}x^j$.

### 4.3. Frame one-forms

We have defined one-forms $\hat{\xi}^\mu$ with vector-like transformation behaviour under $SO_a(n)$. Alternatively we can define one-forms starting from the condition that they commute with coordinates $[\hat{\omega}^\mu, \hat{x}^\nu] = 0$ and therefore with all functions. We make the ansatz

$$\begin{aligned}\hat{\xi}^n &= \hat{\omega}^n g_1(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^j \hat{D}_j g_2(\hat{D}_n, \hat{D}_l\hat{D}_l),\\ \hat{\xi}^i &= \hat{\omega}^n \hat{D}_i h_1(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^i h_2(\hat{D}_n, \hat{D}_l\hat{D}_l) + \hat{\omega}^j \hat{D}_j \hat{D}_i h_3(\hat{D}_n, \hat{D}_l\hat{D}_l),\end{aligned} \tag{58}$$

with functions of the Dirac derivative with appropriate index structure and expand (48). Since we assume that $\hat{\omega}^\mu$ commute with the coordinates, we can reduce the result to commutators of the functions of derivatives $g_a$ and $h_a$ with the coordinates. With the abbreviations

$$\zeta_1 = \frac{1}{\sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}, \qquad \zeta_2 = \frac{1 - \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}{\hat{D}_\sigma\hat{D}_\sigma}, \qquad \zeta_3 = -\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}, \tag{59}$$

and the identities

$$\frac{\partial}{\partial\hat{D}_n}\zeta_2 = \hat{D}_n\zeta_1\zeta_2^2, \qquad \frac{\partial}{\partial\hat{D}_j\hat{D}_j}\zeta_2 = \frac{1}{2}\zeta_1\zeta_2^2, \qquad \frac{\partial}{\partial\hat{D}_n}\zeta_3 = -\mathrm{i}a\zeta_1\zeta_3, \tag{60}$$

equation (48) can be rewritten as a system of differential equations for $g_1, g_2, h_1, h_2$ and $h_3$. Its solution is

$$\begin{aligned} g_1 &= \left(1 + \hat{D}_j\hat{D}_j\zeta_2\zeta_3^{-1}\right)\frac{a^2}{2}\zeta_2, & h_1 &= (-\mathrm{i}a - \hat{D}_n\zeta_2)\frac{a^2}{2}\zeta_2\zeta_3^{-1},\\ g_2 &= (\mathrm{i}a + \hat{D}_n\zeta_2)\frac{a^2}{2}\zeta_2\zeta_3^{-1}, & h_2 &= \frac{a^2}{2}\zeta_2,\\ & & h_3 &= \frac{a^2}{2}\zeta_2^2\zeta_3^{-1}.\end{aligned} \tag{61}$$

Writing the differential d in terms of the frame one-forms $\hat{\omega}^\mu$ we obtain

$$\begin{aligned}\mathrm{d} = \hat{\xi}^\mu\hat{D}_\mu &= \left(\hat{\omega}^n\hat{D}_n - \mathrm{i}a\hat{\omega}^n\hat{D}_l\hat{D}_l\zeta_3^{-1} + \hat{\omega}^j\hat{D}_j\zeta_3^{-1}\right)\frac{a^2}{2}\zeta_2\\ &= \left(\hat{\omega}^n\hat{D}_n + \frac{\hat{\omega}^j\hat{D}_j - \mathrm{i}a\hat{\omega}^n\hat{D}_l\hat{D}_l}{-\mathrm{i}a\hat{D}_n + \sqrt{1 - a^2\hat{D}_\mu\hat{D}_\mu}}\right)\frac{a^2}{2}\frac{1 - \sqrt{1 - a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}.\end{aligned} \tag{62}$$

We can simplify this result using the Laplace operator $\hat{\Box}$ and the derivatives $\hat{\partial}_\mu$

$$\mathrm{d} = \left(\hat{\omega}^n\left(\frac{1}{a}\sin(a\hat{\partial}_n) - \frac{\mathrm{i}a}{2}\hat{\partial}_l\hat{\partial}_l\,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\right) + \hat{\omega}^j\hat{\partial}_j\right)\frac{1}{1 - \frac{a^2}{4}\hat{\Box}}. \tag{63}$$

To determine the transformation behaviour of $\hat{\omega}^\mu$ under $SO_a(n)$-rotations, we first consider the derivative operators dual to $\hat{\omega}^\mu$. The factor $\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$ is an invariant under $SO_a(n)$-rotations by itself. We define

$$\widetilde{\partial}_j = \hat{\partial}_j, \qquad \widetilde{\partial}_n = \frac{1}{a}\sin(a\hat{\partial}_n) - \frac{\mathrm{i}a}{2}\hat{\partial}_j\hat{\partial}_j\,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}. \tag{64}$$

By means of (17) we can calculate

$$\begin{aligned}&[M^{rs}, \widetilde{\partial}_j] = \delta^r_j\widetilde{\partial}_s - \delta^s_j\widetilde{\partial}_r,\\ &[M^{rs}, \widetilde{\partial}_n] = 0,\\ &[N^l, \widetilde{\partial}_j] = -\delta^l_j\widetilde{\partial}_n\sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} - \mathrm{i}a\delta^l_j\widetilde{\partial}_\mu\widetilde{\partial}_\mu + \mathrm{i}a\widetilde{\partial}_j\widetilde{\partial}_l,\\ &[N^l, \widetilde{\partial}_n] = \widetilde{\partial}_l(\mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}).\end{aligned} \tag{65}$$

The derivatives $\widetilde{\partial}_\mu$ are a module of $SO_a(n)$ and $[N^l, \widetilde{\partial}_\mu\widetilde{\partial}_\mu] = 0$. Comparing with (29) we find

$$\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} = \frac{-\mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1 - a^2\widetilde{\partial}_k\widetilde{\partial}_k}, \qquad \mathrm{e}^{\mathrm{i}a\hat{\partial}_n} = \mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}, \tag{66}$$

and the coproducts are

$$\begin{aligned}
&\widetilde{\partial}_j(\hat{f}\cdot\hat{g}) = \widetilde{\partial}_j(\hat{f})\cdot\hat{g} + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\widetilde{\partial}_j\hat{g}),\\
&\widetilde{\partial}_n(\hat{f}\cdot\hat{g}) = \widetilde{\partial}_n(\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\hat{g}) + (\mathrm{e}^{\mathrm{i}a\hat{\partial}_n}\hat{f})\cdot(\widetilde{\partial}_n\hat{g}) - \mathrm{i}a(\widetilde{\partial}_k\hat{f})\cdot(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\widetilde{\partial}_k\hat{g}).
\end{aligned} \tag{67}$$

For compactness, we have used (66) to write (67). We find $\widetilde{\partial}_\mu\widetilde{\partial}_\mu = \hat{D}_\mu\hat{D}_\mu$. Therefore the Laplace operator $\hat{\Box}$ can be written in terms of $\widetilde{\partial}_\mu$ as $\hat{\Box} = \frac{2}{a^2}(1 - \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu})$. Note that $\widetilde{\partial}_\mu = S(\hat{D}_\mu)$, the antipode of the Dirac derivative. Thus, we can write (63) purely in terms of $\hat{\omega}^\mu$ and $\widetilde{\partial}_\mu$:

$$\mathrm{d} = (\hat{\omega}^n\widetilde{\partial}_n + \hat{\omega}^j\widetilde{\partial}_j)\frac{2}{1 + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}. \tag{68}$$

Comparing with (51), we can now evaluate the action of the differential on an arbitrarily ordered one-form

$$\mathrm{d}\alpha = \mathrm{d}(\alpha_\mu(\hat{x})\hat{\omega}^\mu) = \left(\frac{2\widetilde{\partial}_\nu}{1 + \sqrt{1 - a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}\alpha_\mu(\hat{x})\right)\hat{\omega}^\nu\hat{\omega}^\mu. \tag{69}$$

From (65) we can calculate the transformation behaviour of $\hat{\omega}^\mu$ from the requirement that d is an invariant:

$$\begin{aligned}
&[M^{rs}, \hat{\omega}^n] = 0, && [N^l, \hat{\omega}^n] = \hat{\omega}^l\sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + \mathrm{i}a(\hat{\omega}^l\widetilde{\partial}_n - \hat{\omega}^n\widetilde{\partial}_l),\\
&[M^{rs}, \hat{\omega}^j] = \delta^{rj}\hat{\omega}^s - \delta^{sj}\hat{\omega}^r, && [N^l, \hat{\omega}^j] = -\delta^{lj}\hat{\omega}^n\sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu} + \mathrm{i}a(\hat{\omega}^l\widetilde{\partial}_j - \hat{\omega}^j\widetilde{\partial}_l).
\end{aligned}$$

The frame one-forms form a module of $SO_a(n)$ rotations.

The commutation relations between derivatives $\widetilde{\partial}_\mu$ and coordinates are

$$\begin{aligned}
&[\widetilde{\partial}_j, \hat{x}^n] = \mathrm{i}a\widetilde{\partial}_j, && [\widetilde{\partial}_n, \hat{x}^n] = \frac{-\mathrm{i}a^3\widetilde{\partial}_s\widetilde{\partial}_s\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1 - a^2\widetilde{\partial}_k\widetilde{\partial}_k},\\
&[\widetilde{\partial}_j, \hat{x}^i] = \delta^i_j, && [\widetilde{\partial}_n, \hat{x}^i] = -\mathrm{i}a\widetilde{\partial}_i\frac{-\mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}{1 - a^2\widetilde{\partial}_k\widetilde{\partial}_k}.
\end{aligned} \tag{71}$$

Taking into account $\frac{2}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}$ in the commutator with the coordinates, we define $\eth_\nu = \frac{2\widetilde{\partial}_\nu}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}$ as the derivatives dual to $\hat{\omega}^\mu$. Using $\frac{2}{1+\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}} = 1 + \frac{a^2}{4}\eth_\mu\eth_\mu$, we obtain

$$[\eth_j, \hat{x}^n] = \frac{\mathrm{i}a}{2}\eth_j\left(1 + \frac{a^2}{4}\eth_\mu\eth_\mu\right)\left(1 + \frac{\left(1 - \mathrm{i}a\eth_n - \frac{a^2}{4}\eth_\nu\eth_\nu\right)\left(1 + \frac{a^2}{4}\eth_\lambda\eth_\lambda\right)}{1 + \frac{a^2}{4}\eth_\rho\eth_\rho - a^2\eth_k\eth_k}\right),$$

$$[\eth_j, \hat{x}^i] = \delta^i_j\left(1 + \frac{a^2}{4}\eth_\mu\eth_\mu\right) + \frac{a^2}{2}\eth_i\eth_j\cdot\frac{1 - \mathrm{i}a\eth_n - \frac{a^2}{4}\eth_\nu\eth_\nu}{1 + \frac{a^2}{4}\eth_\rho\eth_\rho - a^2\eth_k\eth_k},$$

$$
\begin{aligned}
[\check{\eth}_n, \hat{x}^i] &= -\frac{\mathrm{i}a}{2}\check{\eth}_i\left(1 + \frac{1 - \frac{a^2}{4}\check{\eth}_\nu\check{\eth}_\nu}{1 + \frac{a^2}{4}\check{\eth}_\lambda\check{\eth}_\lambda} + \frac{\mathrm{i}a}{2}\check{\eth}_n\frac{1}{1 + \frac{a^2}{4}\check{\eth}_\rho\check{\eth}_\rho}\right), \\
[\check{\eth}_n, \hat{x}^n] &= \left(1 + \frac{a^2}{4}\check{\eth}_\mu\check{\eth}_\mu\right)\cdot\left(1 + \frac{-\frac{\mathrm{i}a^3}{8}\check{\eth}_s\check{\eth}_s\check{\eth}_n + \left(1 - \frac{a^2}{4}\check{\eth}_\mu\check{\eth}_\mu\right)\left(1 + \frac{a^2}{4}\check{\eth}_\nu\check{\eth}_\nu\right)^2}{\left(1 + \frac{a^2}{4}\check{\eth}_\rho\check{\eth}_\rho\right)^2\left(1 + \left(\frac{a^2}{4}\check{\eth}_\sigma\check{\eth}_\sigma\right)^2 + \frac{a^2}{2}(\check{\eth}_n\check{\eth}_n - \check{\eth}_k\check{\eth}_k)\right.}\right).
\end{aligned}
\tag{72}
$$

These complicated commutators are the price we have to pay for the fact that frame one-forms commute with all functions.

### 4.4. Volume form

The result (48) allows us to calculate the commutation relations of higher-order forms $\hat{\xi}^{\mu_1}\cdots\hat{\xi}^{\mu_j}$ with the coordinates. We can therefore calculate two-forms, three-forms etc up to $n$-forms, the full Hodge differential calculus.

Since we know that the $\hat{\xi}^\mu$ anti-commute among themselves, the dimension of the set of $j$-forms is $\binom{n}{j}$. From the vector-like transformation behaviour of $\hat{\xi}^\mu$ (45) follows that $j$-forms transform as $j$-tensors. Specifically, there is only one $n$-form $\hat{\xi}^1\hat{\xi}^2\cdots\hat{\xi}^n$, which should be a noncommutative analogue of the volume form. It has particularly simple properties; from (48) we can calculate the commutator

$$
[\hat{\xi}^1\hat{\xi}^2\cdots\hat{\xi}^n, \hat{x}^\mu] = n\hat{\xi}^1\hat{\xi}^2\cdots\hat{\xi}^n\hat{D}_\mu\frac{1 - \sqrt{1 - a^2\hat{D}_\sigma\hat{D}_\sigma}}{\hat{D}_\lambda\hat{D}_\lambda}. \tag{73}
$$

The volume form $\hat{\xi}^1\hat{\xi}^2\cdots\hat{\xi}^n$ is invariant under $SO_a(n)$: $[M^{\mu\nu}, \hat{\xi}^1\hat{\xi}^2\cdots\hat{\xi}^n] = 0$.

In contrast to this result, the $n$-form built from $n$ different frame one-forms $\hat{\omega}^1\hat{\omega}^2\cdots\hat{\omega}^n$ is not invariant under $SO_a(n)$: $[N^l, \hat{\omega}^1\hat{\omega}^2\cdots\hat{\omega}^n] = -\mathrm{i}a(n-1)\hat{\omega}^1\hat{\omega}^2\cdots\hat{\omega}^n\hat{\partial}_l$.

The $\star$-representation of the volume form is

$$
(\xi^1\xi^2\cdots\xi^n)^* = \frac{\mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n}{\left(1 + \frac{\partial_\mu\partial_\mu}{2\partial_n^2}(\cos(a\partial_n) - 1)\right)^n}, \tag{74}
$$

while the representation of the frame $n$-form is simply $(\omega^1\omega^2\cdots\omega^n)^* = \mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n$.

## 5. Vector fields

### 5.1. Linearly transforming vector fields and conjugation

The aim of our work is to define physical field theories. Vector fields that have the same transformation properties as derivatives under $SO_a(n)$ are a necessary ingredient for the definition of gauge theories.

The central assumption is that the transformation behaviour of vector fields is such that the vector fields appear linearly on the right-hand side of the commutation relations.

Vector fields analogous to the vector-like transforming Dirac derivative are obviously

$$
\begin{aligned}
&[M^{rs}, \hat{V}_n] = 0, \qquad && [M^{rs}, \hat{V}_i] = \delta_i^r\hat{V}^s - \delta_i^s\hat{V}^r, \\
&[N^l, \hat{V}_n] = \hat{V}^l, \qquad && [N^l, \hat{V}_i] = -\delta_i^l\hat{V}_n,
\end{aligned}
\tag{75}
$$

these vector fields $\hat{V}_\mu$ are a module of $SO_a(n)$ rotations.

It is more difficult to find vector fields with transformation properties analogous to the other derivatives that we have defined throughout this paper. Although we have argued that the derivatives $\hat{\partial}_\mu$ are in a sense irrelevant for the geometric construction of $\kappa$-deformed space, they have an important role to play in making contact with the commutative regime. Since $\hat{\partial}_n = \partial_n$ on all three $\star$-products, these derivatives $\hat{\partial}_\mu$ provide information on the connection between the abstract algebra and $\star$-product representation. Therefore we now investigate vector fields $\hat{A}_\mu$ analogous to $\hat{\partial}_\mu$. By this we mean that we construct the tramsformation law of $\hat{A}_\mu$ in such a way that it conicides with (17), when $\hat{A}_\mu$ is re-substituted with $\hat{\partial}_\mu$. At the same time $\hat{A}_\mu$ must be a module of $SO_a(n)$ rotations. We make the choice that derivatives are always to the left of the vector field $\hat{A}_\mu$ in nonlinear expressions such as the vector field analogue of (17). We stress that $\hat{A}_\mu$ is treated as an element of an abstract algebra in this approach. Therefore derivatives are not evaluated on $\hat{A}_\mu$ in terms of the coproduct.

The problem can be solved in a power series expansion in $a$. This results in a recursion formula with the solution[6]:

$$
\begin{aligned}
&[M^{rs}, \hat{A}_i] = \delta_i^r \hat{A}_s - \delta_i^s \hat{A}_r, \qquad [M^{rs}, \hat{A}_n] = 0,\\
&[N^l, \hat{A}_i] = \delta_i^l \frac{1-\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n}}{2\,\mathrm{i}a\hat{\partial}_n}\hat{A}_n - \frac{\mathrm{i}a}{2}\delta_i^l \hat{\partial}_j \hat{A}_j + \frac{\mathrm{i}a}{2}(\hat{\partial}_l \hat{A}_i + \hat{\partial}_i \hat{A}_l)\\
&\qquad - \delta_i^l \frac{a}{2\hat{\partial}_n}\tan\left(\frac{a\hat{\partial}_n}{2}\right)(\hat{\partial}_n\hat{\partial}_j\hat{A}_j - \hat{\partial}_j\hat{\partial}_j\hat{A}_n)\\
&\qquad + \left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\left(\frac{a\hat{\partial}_n}{2}\right)\right)(\hat{\partial}_n\hat{\partial}_i\hat{A}_l + \hat{\partial}_n\hat{\partial}_l\hat{A}_i - 2\hat{\partial}_l\hat{\partial}_i\hat{A}_n),\\
&[N^l, \hat{A}_n] = \hat{A}_l.
\end{aligned} \tag{76}
$$

The square of the vector field corresponding to the Dirac derivative is an invariant $[M^{\mu\nu}, \hat{V}_\lambda \hat{V}_\lambda] = 0$. To form an invariant out of the vector field $\hat{A}_\mu$, we have to define a vector field $\check{A}_\mu$ with transformation laws in which the derivatives are to the right of the vector field $\check{A}_\mu$. We demand

$$
[M^{rs}, \check{A}_\lambda \hat{A}_\lambda] = 0, \qquad \text{and} \qquad [N^l, \check{A}_\lambda \hat{A}_\lambda] = 0. \tag{77}
$$

From (76) we can construct the transformation laws for $\check{A}_\mu$ such that (77) is fulfilled:

$$
\begin{aligned}
&[M^{rs}, \check{A}_i] = \delta_i^r \check{A}_s - \delta_i^s \check{A}_r, \qquad [M^{rs}, \check{A}_n] = 0,\\
&[N^l, \check{A}_i] = -\delta_i^l \check{A}_n + \frac{\mathrm{i}a}{2}\check{A}_l\hat{\partial}_i - \frac{\mathrm{i}a}{2}\check{A}_i\hat{\partial}_l - \frac{\mathrm{i}a}{2}\delta_i^l\check{A}_j\hat{\partial}_j + \frac{a}{2}\check{A}_l \tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}_i\\
&\qquad - \left(\delta_i^l\check{A}_j\hat{\partial}_j + \check{A}_i\hat{\partial}_l\right)\left(\frac{1}{\hat{\partial}_n} - \frac{a}{2}\cot\left(\frac{a\hat{\partial}_n}{2}\right)\right),
\end{aligned} \tag{78}
$$

$$
[N^l, \check{A}_n] = \check{A}_l \frac{\mathrm{e}^{2\mathrm{i}a\hat{\partial}_n} - 1}{2\mathrm{i}a\hat{\partial}_n} - \check{A}_l \frac{a}{2\hat{\partial}_n}\tan\left(\frac{a\hat{\partial}_n}{2}\right)\hat{\partial}_j\hat{\partial}_j + 2\check{A}_j\left(\frac{1}{\hat{\partial}_n^2} - \frac{a}{2\hat{\partial}_n}\cot\left(\frac{a\hat{\partial}_n}{2}\right)\right)\hat{\partial}_l\hat{\partial}_j.
$$

With this transformation law the vector fields $\check{A}_\mu$ are a module of $SO_a(n)$ rotations.

All relations considered up to now are invariant under the conjugation

$$
\begin{aligned}
(\hat{x}^\mu)^\dagger &= \hat{x}^\mu, &\qquad (\hat{\partial}_\mu)^\dagger &= -\hat{\partial}_\mu,\\
(M^{rs})^\dagger &= -M^{rs}, &\qquad (N^l)^\dagger &= -N^l.
\end{aligned} \tag{79}
$$

Comparing (76) and (78), we see that $\check{A}_\mu$ transforms with the derivatives on the right-hand side, but $\check{A}_\mu^\dagger \neq \hat{A}_\mu$, they transform in different ways. The transformation for $\hat{A}_\mu^\dagger$ is simply

[6] This solution is not unique. If the symmetrization in the third term of $[N^l, \hat{A}_i]$ is not performed, the last term of $[N^l, \hat{A}_i]$ vanishes.

(76), with all $\hat{A}_\mu$ standing to the far right in any expression replaced by $\hat{A}^\dagger_\mu$ standing to the far left.

The dual of $\hat{A}^\dagger_\mu$ is $\check{A}^\dagger_\mu$, $\left[N^l, \hat{A}^\dagger_\mu \check{A}^\dagger_\mu\right] = \left[M^{rs}, \hat{A}^\dagger_\mu \check{A}^\dagger_\mu\right] = 0$. The dual vector field $\check{A}^\dagger_\mu$ transforms as in (78), with all $\check{A}_\mu$ standing to the far left in any expression replaced by $\check{A}^\dagger_\mu$ standing to the far right.

### 5.2. Vector fields related to frame one-forms

In the same manner as we derived the vector fields corresponding to $\hat{\partial}_\mu$, we can also calculate vector fields $\widetilde{A}_\mu$, corresponding to $\widetilde{\partial}_\mu$, the derivative dual to the frame one-forms up to the factor $\frac{1}{1-\frac{a^2}{4}\hat{\Box}}$. The calculation is much simpler and we obtain

$$
\begin{aligned}
&[M^{rs}, \widetilde{A}_i] = \delta^r_i \widetilde{A}_s - \delta^s_i \widetilde{A}_r, \qquad [M^{rs}, \widetilde{A}_n] = 0,\\
&[N^l, \widetilde{A}_i] = -\delta^l_i \sqrt{1 - a^2 \widetilde{\partial}_\mu \widetilde{\partial}_\mu}\, \widetilde{A}_n + \mathrm{i}a\widetilde{\partial}_i \widetilde{A}_l - \mathrm{i}a\delta^l_i \widetilde{\partial}_\mu \widetilde{A}_\mu,\\
&[N^l, \widetilde{A}_n] = (\mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2 \widetilde{\partial}_\mu \widetilde{\partial}_\mu})\widetilde{A}_l.
\end{aligned} \tag{80}
$$

From (80) we could read off immediately the transformation behaviour of $\widetilde{A}^\dagger_\mu$, but comparing with $\check{\widetilde{A}}_\mu$, which can be obtained from the invariant

$$
[M^{rs}, \check{\widetilde{A}}_\mu \widetilde{A}_\mu] = 0, \qquad \text{and} \qquad [N^l, \check{\widetilde{A}}_\mu \widetilde{A}_\mu] = 0, \tag{81}
$$

we find that $\widetilde{A}^\dagger_\mu = \check{\widetilde{A}}_\mu$:

$$
\begin{aligned}
&\left[M^{rs}, \widetilde{A}^\dagger_i\right] = \delta^r_i \widetilde{A}^\dagger_s - \delta^s_i \widetilde{A}^\dagger_r, \qquad \left[M^{rs}, \widetilde{A}^\dagger_n\right] = 0,\\
&\left[N^l, \widetilde{A}^\dagger_i\right] = -\delta^l_i \widetilde{A}^\dagger_n \sqrt{1 - a^2 \widetilde{\partial}_\mu \widetilde{\partial}_\mu} + \mathrm{i}a \widetilde{A}^\dagger_l \widetilde{\partial}_i - \mathrm{i}a\delta^l_i \widetilde{A}^\dagger_\mu \widetilde{\partial}_\mu,\\
&\left[N^l, \widetilde{A}^\dagger_n\right] = \widetilde{A}^\dagger_l (\mathrm{i}a\widetilde{\partial}_n + \sqrt{1 - a^2 \widetilde{\partial}_\mu \widetilde{\partial}_\mu}).
\end{aligned} \tag{82}
$$

In this sense, the vector field $\widetilde{A}_\mu$ is self-dual.

### 5.3. Derivative-valued vector fields

We will now show that $\hat{A}_\mu$, $\check{A}_\mu$, $\check{A}^\dagger_\mu$ and $\hat{A}^\dagger_\mu$ can be obtained from the vector field $\hat{V}_\mu$ by a derivative-valued map $\hat{V}_\mu = \hat{e}_{\mu\nu} \hat{A}_\nu$, $\hat{A}_\mu = (\hat{e}^{-1})_{\mu\nu} \hat{V}_\nu$. This is a change in the basis of derivatives, $\hat{e}_{\mu\nu} = \hat{e}_{\mu\nu}(\partial)$ depends on the derivatives.

We know the transformation properties of $\hat{V}_\mu$, $\hat{A}_\mu$ and $\hat{\partial}_\mu$, (75), (76) and (17). We expand these in powers of $a$, at zeroth order we assume that $\hat{V}_\mu|_{\mathcal{O}(a^0)} = \hat{A}_\mu|_{\mathcal{O}(a^0)}$ are the same vector field. We obtain a recursion formula in $a$ that can be solved:

$$
\begin{aligned}
\hat{e}_{nn} &= \frac{1}{a\hat{\partial}_n} \sin(a\hat{\partial}_n) + \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} \left( \frac{\mathrm{i}a}{2} - \frac{\mathrm{i}}{\hat{\partial}_n} \tan\left(\frac{a\hat{\partial}_n}{2}\right)\right) \frac{\hat{\partial}_k \hat{\partial}_k}{\hat{\partial}_n},\\
\hat{e}_{nj} &= \frac{\mathrm{i}}{\hat{\partial}_n} \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} \tan\left(\frac{a\hat{\partial}_n}{2}\right) \hat{\partial}_j,\\
\hat{e}_{ln} &= \left( \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n} - \frac{1 - \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}}{\mathrm{i}a\hat{\partial}_n}\right) \frac{\hat{\partial}_l}{\hat{\partial}_n},\\
\hat{e}_{lj} &= \frac{1 - \mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}}{\mathrm{i}a\hat{\partial}_n} \delta_{lj}.
\end{aligned} \tag{83}
$$

To find the inverse of the matrix $\hat{e}_{\mu\nu}$, we have to take care to single out the right partial derivatives. The result is

$$
\begin{aligned}
(\hat{e}^{-1})_{nn} &= F^{-1}(\hat{\partial}_\mu)\frac{\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-1}{-\mathrm{i}a\hat{\partial}_n},\\
(\hat{e}^{-1})_{nj} &= F^{-1}(\hat{\partial}_\mu)\left(-\frac{\mathrm{i}}{\hat{\partial}_n}\,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\tan\left(\frac{a\hat{\partial}_n}{2}\right)\right)\hat{\partial}_j,\\
(\hat{e}^{-1})_{ln} &= F^{-1}(\hat{\partial}_\mu)\left(\frac{\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-1}{-\mathrm{i}a\hat{\partial}_n}-\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\right)\hat{\partial}_l,\\
(\hat{e}^{-1})_{lj} &= \frac{-\mathrm{i}a\hat{\partial}_n}{\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-1}\delta_{lj}+\frac{\frac{\mathrm{i}}{\hat{\partial}_n^2}\,\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}\tan\left(\frac{a\hat{\partial}_n}{2}\right)\left(\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-\frac{\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-1}{-\mathrm{i}a\hat{\partial}_n}\right)}{F(\hat{\partial}_\mu)\left(\frac{\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}-1}{-\mathrm{i}a\hat{\partial}_n}\right)}\hat{\partial}_l\hat{\partial}_j,\\
F(\hat{\partial}_\mu) &= \left(\frac{1}{\mathrm{i}a^2\hat{\partial}_n^2}\sin(a\hat{\partial}_n)(1-\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n})-\frac{\hat{\partial}_k\hat{\partial}_k}{2\mathrm{i}\hat{\partial}_n^2}\tan\left(\frac{a\hat{\partial}_n}{2}\right)\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n}(1-\mathrm{e}^{-\mathrm{i}a\hat{\partial}_n})\right).
\end{aligned}
\tag{84}
$$

The vector field $\check{A}_\mu$ is also defined by its transformation behaviour that was derived from (77). As the derivatives are on the right of $\check{A}_\mu$ we make the ansatz $\check{A}_\nu = \hat{V}_\mu\big(\check{e}^{-1}_{\mu\nu}\big)$.

This ansatz is inserted into $\check{A}_\mu\hat{A}_\mu = \hat{V}_\rho(\check{e}^{-1})_{\rho\mu}(\hat{e}^{-1})_{\mu\nu}\hat{V}_\nu$. But we know that $\hat{V}_\mu\hat{V}_\mu$ is an invariant, therefore we conclude that $(\check{e}^{-1})_{\rho\mu}(\hat{e}^{-1})_{\mu\nu} = \delta_{\rho\nu}$, which leads to

$$(\check{e}^{-1})_{\rho\mu} = \hat{e}_{\rho\mu}. \tag{85}$$

The formulae for $\check{A}^\dagger_\mu$ and $\hat{A}^\dagger_\mu$ are obtained by conjugation.

There is also a transformation matrix $\widetilde{e}_{\mu\nu}$ from $\hat{V}_\mu$ to $\widetilde{A}_\nu$ (respectively $\widetilde{\check{A}}_\nu = \widetilde{A}^\dagger_\nu$) :

$$\widetilde{A}_\mu = \widetilde{e}_{\mu\nu}\hat{V}_\nu, \qquad \widetilde{A}^\dagger_\mu = \hat{V}_\nu\widetilde{e}_{\mu\nu}, \tag{86}$$

which is

$$
\begin{aligned}
\widetilde{e}_{nn} &= 1, &\qquad \widetilde{e}_{nj} &= -\mathrm{i}a\hat{D}_j\frac{\mathrm{i}a\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k}\\
\widetilde{e}_{ln} &= \mathrm{i}a\hat{D}_l, &\qquad \widetilde{e}_{lj} &= \delta_{lj}\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}+a^2\hat{D}_j\hat{D}_l\frac{\mathrm{i}a\hat{D}_n+\sqrt{1-a^2\hat{D}_\mu\hat{D}_\mu}}{1-a^2\hat{D}_k\hat{D}_k}.
\end{aligned}
\tag{87}
$$

The inverse matrix $\hat{V}_\mu = \widetilde{A}^\dagger_\nu(\widetilde{e}^{-1})_{\mu\nu}$, with $\widetilde{e}_{\nu\mu}(\widetilde{e}^{-1})_{\lambda\nu} = \delta_{\mu\lambda}$ is

$$
\begin{aligned}
(\widetilde{e}^{-1})_{nn} &= 1+\frac{a^2\widetilde{\partial}_k\widetilde{\partial}_k}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}\frac{-\mathrm{i}a\widetilde{\partial}_n+\sqrt{1-a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}{1-a^2\widetilde{\partial}_s\widetilde{\partial}_s}, &\qquad (\widetilde{e}^{-1})_{nj} &= \frac{\mathrm{i}a\widetilde{\partial}_j}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}},\\
(\widetilde{e}^{-1})_{ln} &= \frac{-\mathrm{i}a\widetilde{\partial}_l}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}\frac{-\mathrm{i}a\widetilde{\partial}_n+\sqrt{1-a^2\widetilde{\partial}_\lambda\widetilde{\partial}_\lambda}}{1-a^2\widetilde{\partial}_k\widetilde{\partial}_k}, &\qquad (\widetilde{e}^{-1})_{lj} &= \delta_{lj}\frac{1}{\sqrt{1-a^2\widetilde{\partial}_\mu\widetilde{\partial}_\mu}}.
\end{aligned}
\tag{88}
$$

## 6. Conclusion

In this work we have shown how to construct algebraic–geometric quantities on a specific noncommutative space, the $\kappa$-deformed space. This method allows us to define algebraic–geometric quantities via their consistency with the defining relations of the noncommutative space, adding a minimal set of additional requirements. For example, we have shown how to construct differential forms by demanding consistency with the coordinate algebra and a specific transformation behaviour under $SO_a(n)$ rotations.

The method presented here does not require a thorough understanding of deep mathematical concepts such as Hopf algebras, for treating noncommutative spaces. The Hopf algebra description of the $SO_a(n)$ symmetry, however, can be fully recaptured in this context.

For more general noncommutative spaces than the $\kappa$-deformed space, our approach might not automatically lead to well-founded results. However, since this method has shown so fruitful here, leading directly to workable definitions of derivatives, differential forms and vector fields, we suggest that the presented method can be used also to investigate other noncommutative spaces.

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